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Conjectural Approach to Pole Placement

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***Abstract*—** A method for pole assignment is developed following a conjecture arising from the observation of assigning a single pole in the case of a one-state system and/or a single pole for a single input system. The method is first justified for extreme cases, and then proved for the general case. The method does not rely on state transformation; and applies to multi-input systems equally well. The uncontrollable case is also considered, with subsequent simplifying relationships, where in certain cases the matrix inverse involved in computations reduces any left inverse of the input matrix.

***Keywords*—** Conjectures; Pole placement; Eigenvalue assignment; State feedback; Entire eigenstructure assignment.

1. INTRODUCTION

Pole assignment and eigenvalue assignment are the same. They are needed to stabilize systems, control the speed of response, and together with the eigenvectors shape the system's response. The literature is full of so many methods, each offering its own advantageous features.

Since we are focusing on a particular conjectural approach and have no intention to delve into the vast area of research on pole assignment, we refer the reader to some books on the subject matter [1, 2], in addition to adequate references quoted below.

In a cursory account, the problem of pole assignment has been tackled through many approaches ranging from geometric methods [3], dyadic based methods [1, 2], eigenstructure methods [4-7], explicit determination [8], recursive methods [9], closed loop robustness [10], minimization of certain condition numbers [11], and use of upper block Heisenberg forms which use numerically preferable orthogonal transformation [12-23], to mention but a few.

Most methods do not require knowledge of the closed loop eigenvectors. A design option is worth consideration in some applications involving robustness [10] and system response shaping [4, 5]. In discrepancy to the entire eigenstructure method [4-7] that requires all n closed loop eigenvectors, the method described in this paper requires some of these eigenvectors depending on the rank m of the input matrix B.

A hyperplane design of variable structure systems fits within the general problem of state feedback as a special case. Therefore, pole assignment can benefit from such methods that are specific to the hyperplane design. Zinober [24] has provided a CAD VSC Toolbox written in MATLAB programming environment. His design approaches were based on theories of quadratic performance minimization, eigenstructure assignment, sensitivity reduction, eigenvalue assignment within a sector, disc, and vertical strip. Such notions help fulfill requirements of system specifications like: settling time, rise time, maximum overshoot and steady state errors.

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Recent applications of the pole placement are in civil structures subjected to earthquake excitation [25], and to second order systems [26]. Besides, pole placement has been implemented through a derivative and acceleration feedback [27, 28].

In order to carry out the assignment process, a maximum of n - m closed loop eigenvectors is needed. In this respect, the method is partly based on eigenvalue–eigenvector assignment. The first introduction of eigenvalue - eigenvector assignment in shaping system transient response has been given in [4]. The technique is a subject of an entire book by Liuet et al., [18]; and has been surveyed by White [19].

Throughout the history of science, conjectures have always been a starting point for many theorems and scientific facts. Such a conception is considered in this work.

The purpose of our study is to prove a conjecture regarding an assignment law based on an observation concerning the assignment of a single eigenvalue for the trivial case $n = 1$, i.e. a single state system, and then to the case of assigning a single eigenvalue when $n > 1$ and $m = 1$. The extreme case where $m = n$, and the nonsingular input matrix further enhances the conjecture.

The application of the method does not distinguish between controllable and uncontrollable systems, between single-input or multi-input systems. It does not need state transformation; and is applicable to a repeated and complex eigenvalues assignment.

The derivation ends up with a solution of two sets of matrix equations. The extension of the method to uncontrollable systems is straightforward; and it offers some cut down measures in the calculations involved.

2. BASIS OF THE CONJECTURE

The terms poles and eigenvalues are the same since the poles of a transfer function are the eigenvalues of the system A matrix. In addition, the assignment and placement terms are used interchangeably.

The pole placement control law considered in this paper is a state feedback law of the form $u = K\,x$ applied to the system:

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in R^n, u \in R^m$ and B have full rank m. The closed loop system is therefore given by:

$$\dot{x} = Ax + BK\,x = (A + BK)\ x \tag{2}$$

The assignment problem seeks convenient methods for the calculation of K; the control literature teems with countless methods, which result in an assignment of n desirable poles.

Among the myriad methods is a method utilizing eigenvectors known as the entire eigenstructure assignment method [5-7]. With this method, the closed loop eigenvalue-eigenvector pair λ_i , w_i for $i = 1$ to n has to satisfy the assignment condition as formulated in [5, 18], i.e. satisfying the following condition:

$$[A - \lambda_i I_n \quad B]\begin{bmatrix} w_i \\ z_i \end{bmatrix} = 0 \ ; \quad \text{where} \quad z_i = K\,w_i \tag{3}$$

To avoid determination of null spaces, as the classic entire eigenstructure method requires, the solution problem is formulated as a solution of a system of linear equations:

$$(A - \lambda_i I_n)\, w_i = -B z_i \tag{4}$$

The solutions obtained are dependent on Z_i which takes arbitrary values. As established by the theories of linear algebra, when λ_i is an eigenvalue of A, more than one solution may exist for w_i. This adds to the flexibility of the method.

The reasoning behind the conjectural approach goes like this:
Suppose we are to assign a single eigenvalue (pole) λ to a scalar system i.e. $n = 1$, then A, B, λ, and w are all scalars represented as $a, b, \lambda,$ and w such as:

$$a + bK = \lambda$$

$$K = b^{-1}(\lambda - a) = b^{-1}\lambda - b^{-1}a \tag{5}$$

or formulated using the determinant method as:

$$\left| s I_n - A - BK \right| = \psi(s) \tag{6}$$

where $\psi(s)$ is the closed loop characteristic equation. For our special case, Eq. (6) is reduced to:

$$\left| s - a - bK \right| = s - \lambda \quad \Rightarrow \quad K = b^{-1}\lambda - b^{-1}a \tag{7}$$

The above form also shows up when considering some well-known methods for the pole assignment such as the Ackermann's method [23].

$$K = -[0 \quad 0 \quad \ldots \quad 1]\left[B \quad AB \quad \ldots \quad A^{n-1}B\right]^{-1} \Psi(A) \tag{8}$$

where $\Psi(A)$ is the desirable closed loop characteristic equation evaluated at A. When adapted to the single-state case, it results in the following K:

$$K = -[1][b]^{-1}(a - \lambda) = b^{-1}\lambda - b^{-1}a \tag{9}$$

It also shows up when adapting the entire eigenstructure method to the single state case, so:

$$(a - \lambda)\, w + bKw = 0$$

$$(a - \lambda)\, w + b\, z = 0$$

$$\Rightarrow z = (b^{-1}\lambda - b^{-1}a)\, w, \text{ where } K = z\, w^{-1} \tag{10}$$

which results in the following:

$$K = b^{-1}\lambda - b^{-1}a \tag{11}$$

Furthermore, suppose now, we are to assign only a single eigenvalue to a system where $n > 1$ and $m = 1$, i.e. w_1 is not now a scalar, but a non-zero vector, so:

$$(A + BK)\, w_1 = \lambda_1 w_1 \tag{12}$$

or

$$BK w_1 = (\lambda_1 I_n - A)\, w_1 \tag{13}$$

What distinguishes this case is that B has no reciprocal. To facilitate a solution and avoid using generalized inverses of matrices, we premultiply Eq. (13) by an m x n matrix G such that GB is nonsingular, ending with:

$$K = (GB)^{-1}(\lambda_1 G - GA) = (GB)^{-1}\lambda_1 G - (GB)^{-1} GA \tag{14}$$

This equation has the same form as that of Eq. (11). But here, $(b)^{-1}$ is replaced by $(GB)^{-1}$. As a matter of fact, $(GB)^{-1}G$ is effectively a left inverse of B. Attempting to assign more than one eigenvalue, we have to solve for:

$$(A+BK)\, w_i = \lambda_i\, w_i \qquad i = 1,\ 2,\ \ldots.,n \tag{15}$$

when Eq. (15) is put in a matrix form, it has to assume the following form:

$$(A+BK)\ W = W\Lambda$$

$$BKW = W\Lambda - AW \tag{16}$$

where Λ is an n x n matrix in a block matrix form specifying the n eigenvalues to be assigned; and W is an n x n matrix containing the associated closed loop eigenvectors.

For the assignment of more than one eigenvalue, Eq. (16) does not permit a solution for the form given in Eq. (14) as multiplication by G does not result in an isolation of K as W cannot be factored out. So, in order to facilitate a solution in the case where $m > 1$ and/or W is a multi-column matrix, a need arises to conjure a solution and prove its validity. This is the main theme of this paper as discussed below.

3. VALIDATION AND PROOF

The observations leads to the following conjecture stated as a theorem:

Theorem 3.1:

Given M and G as m x m and m x n matrices, the following feedback matrix:

$$K = (GB)^{-1}\left(MG - GA\right) \tag{17}$$

assigns m eigenvalues specified by M, and the remaining $n - m$ eigenvalues through a particular choice of G.

Proof:

The conjecture has been validated for the single state case where $n > 1$ together with a single pole assignment case.

It is also beneficial to validate our conjecture for the other extreme case of assigning n eigenvalues when B is square and nonsingular i.e. its rank is m which is also equal to n. In this case the whole set of the desired eigenvalues are those of n x n matrix Λ.

$$(A+BK) = \Lambda \tag{18}$$

Both sides of Eq. (18) are premultiplied by a square nonsingular matrix G. Since both B and G are nonsingular, their product GB is nonsingular, giving:

$$K = (GB)^{-1}(G\Lambda - GA) = (GB)^{-1}(G\Lambda G^{-1} G - GA) \tag{19}$$

Without loss of generality, the matrix $G\Lambda G^{-1}$ can be re-named as M as the two matrices have the same set of eigenvalues. Hence, we obtain Eq. (17) and the conjecture is validated for this extreme special case as well.

Although one may argue that since B is nonsingular, why not to use $K = B^{-1}(\Lambda - A)$ or $K = B^{-1}(M - A)$ instead. This is true, and it is even more explicit, but greater flexibility may be gained when premultiplied by G, especially since we know that for the multi-input case it is well known that the feedback matrix is not unique. In other words, we get a much broader class for *K*.

It remains to prove the conjecture for the general case where *B* is multi-input and non-square, i.e. the assignment of *n* eigenvalues when *B* is a *n* x *m* matrix. To do that, decompose the eigenvalues and eigenvectors into two sets: *m* eigenvalues specified through Λ_m and *n* - *m* eigenvalues specified through Λ_{n-m}, in conjunction with their associated eigenvectors W_m, and W_{n-m} respectively which are obtained using Eq. (4);

$$W_n = [W_m \quad W_{n-m}] \quad , \text{ and } \quad \Lambda_n = \begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix} \tag{20}$$

Substituting K as in Eq. (17), we get:

$$(A+BK)W_n = W_n\Lambda_n = (A+B(GB)^{-1}(MG-GA))[W_m \; W_{n-m}] = [W_m \; W_{n-m}]\begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix} \tag{21}$$

Premultiplying both sides of Eq. (20) by G, the can be simplified as follows:

$$G(A+B(GB)^{-1}(MG-GA))[W_m \; W_{n-m}] = G[W_m \; W_{n-m}]\begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix}$$

$$(GA+GB(GB)^{-1}(MG-GA))[W_m \; W_{n-m}] = [GW_m \; GW_{n-m}]\begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix}$$

$$(GA+I_n(MG-GA))[W_m \; W_{n-m}] = [GW_m \; GW_{n-m}]\begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix}$$

$$(\cancel{GA}+MG-\cancel{GA}))[W_m \; W_{n-m}] = [GW_m \quad GW_{n-m}]\begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix}$$

By invoking the product of partitioned (block) matrices,

$$(MG)[W_m \quad W_{n-m}] = [GW_m \vdots GW_{n-m}]\begin{bmatrix} \Lambda_m & \vdots & 0_{m\times n-m} \\ \cdots & \cdots & \cdots \\ 0_{n-m\times m} & \vdots & \Lambda_{n-m} \end{bmatrix}$$

$$[MGW_m \qquad MGW_{n-m}] = [GW_m\Lambda_m \qquad GW_{n-m}\Lambda_{n-m}] \tag{22}$$

Hence,

$$M\,GW_m = GW_m\,\Lambda_m \tag{23}$$

and

$$M\,GW_{n-m} = GW_{n-m}\,\Lambda_{n-m} \tag{24}$$

we have two sets of matrix equations where *G*, and *M* are to be determined.

It is recalled that *G* is a *m* x *n* matrix and *GB* has to be a *m* x *m* square matrix and invertible. Such a requirement imposes the only choice of letting:

$$GW_{n-m} = 0 \tag{25}$$

or

$$G = W_{n-m}{}^{\perp} \tag{26}$$

That is, G is a non-unique left annihilator of W_{n-m} [20-22]. If instead, we take $GW_m = 0$, then, G will be an $(n - m)$ x n matrix. Consequently GB will not be square unless $n = 2m$.

Any G, which satisfies Eq. (25), results in the assignment of $n - m$ eigenvalues whose eigenvectors are W_{n-m}. It is also important to point out that the m eigenvectors W_m associated with the m eigenvalues are necessarily independent of W_{n-m}. Therefore, W_m and W_{n-m} constitute a basis for R^n and since G annihilates W_{n-m}, it can never annihilate W_m, i.e. the m x m GW_m product is a nonsingular square matrix, hence invertible. This fact is needed to find a solution for Eq. (23) getting:

$$M = (GW_m)\ \Lambda_m\ (GW_m)^{-1} \tag{27}$$

The choice of G as in Eq. (26) and M as in Eq. (27) results in an assignment of a total of n eigenvalues together with their associated eigenvectors. A corollary is now in order.

Corollary 3.1:

m eigenvalues are assigned through M irrespective of G and W.

Proof:

Recall that M, GW_m, and Λ_m are all square matrices. Since matrices M and Λ_m are related through a similarity transformation, they have the same set of eigenvalues. The m eigenvalue of M is the m eigenvalues of Λ_m.

Furthermore, let $\lambda\left[M\right]$ stands for the eigenvalues of M. The following fact regarding the product of matrices is invoked:

$$\lambda\left[M\right] = \lambda\left[(GW_m)\ \Lambda_m\ (GW_m)^{-1}\right] = \lambda\left[\ \Lambda_m\ (GW_m)^{-1}(GW_m)\right] = \lambda\left[\ \Lambda_m\ \right] \tag{28}$$

G, and W_m do not influence the eigenvalues of M. Consequently, an M can be specified regardless of Eq. (27) if the assignment of m eigenvectors is not needed, thus easing the design process.

In other words, the assignment of the m eigenvalues is guaranteed irrespective of G and irrespective of the remaining $n - m$ eigenvalues and associated eigenvectors W_{n-m}. However, according to Eq. (26) a specific G family is still needed to ensure the assignment of the remaining $n-m$ eigenvalues and an invertible GB matrix.

In summary, for controllable systems, $n - m$ eigenvectors are needed to assign $n - m$ eigenvalues with their corresponding W_{n-m} eigenvectors leading to a G matrix that annihilates W_{n-m}. For the remaining m eigenvalues, a family of M matrices with the same eigenvalues as Λ_m does the job irrespective of G and W_m. However, if the associated m eigenvectors are to be also assigned then M is constructed as in Eq. (27). In all cases, the product GB should be nonsingular. For uncontrollable systems, the requirement of the $n - m$ eigenvectors can be relaxed as commented in the following section.

4. UNCONTROLLABLE SYSTEMS

The uncontrollable case requires no additional substantiation. As has been shown above, M assigns any desirable m eigenvalues. However, this ability excludes the assignment of uncontrollable eigenvalues within M. Otherwise, we will have a violation of the proved fact that uncontrollable eigenvalues cannot be changed by state feedback. Therefore, any uncontrollable eigenvalue has to be re-assigned through the $n - m$ set of eigenvectors through G, but not through M. Otherwise, a nonsingular GB cannot be obtained.

A simplifying property arises in the case where we have a maximum number of $n - m$ uncontrollable eigenvalues. An arbitrary G can be chosen in this case provided that a nonsingular GB is ensured. Any left inverse of B can be taken as G for this special case. Besides, an attractive systematic fulfillment is the choice $G = B^T$, which always results in a nonsingular GB equal to BB^T in this special case. BB^T is always nonsingular for full rank B matrices, [22].

It is worth mentioning that although uncontrollable eigenvalues cannot be changed by state feedback, their associated right eigenvectors can provide extra degrees of freedom for shaping the system's response.

5. EXAMPLES

The examples are chosen of low order for the ease of following and rapid checking of the results. We could have used examples with real or irrational numbers as expected when modeling practical systems. However, such an attempt will make following the solution harder with no justification.

Example 1:

Consider the single input uncontrollable system:

$$\dot{x} = \begin{bmatrix} -7 & 3 & 3 \\ -6 & 1 & 4 \\ 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \tag{29}$$

The rank test of controllability reveals that:

$$rank([B \;\; AB \;\; A^2B]) = 2 < n = 3 \tag{30}$$

Hence, the system is uncontrollable. To determine the uncontrollable eigenvalue, we may use the fact that:

$$[A - \lambda I_3 \;\vdots\; B\;] \tag{31}$$

loses rank when λ is an uncontrollable eigenvalue.

The test shows that -1 is an uncontrollable eigenvalue, so it has to be re-assigned. State feedback leaves the uncontrollable eigenvalues invariant, but it permits the re-assignment of the right eigenvectors.

In addition to the -1 re-assignment, it is also desired to assign the two eigenvalues: -2 and -5.

The -1 uncontrollable eigenvalue cannot be reassigned through M but through G. The -5 eigenvalue is chosen to be assigned through G for the sake of exercising.

According to Eq. (4), and in order to get integer values for the eigenvectors we let Z_1 = -2 and Z_2 = -18. The two eigenvectors associated with the -5 and -1 eigenvalues are respectively:

$$w5 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad w1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{32}$$

The annihilating G matrix according to Eq. (26) is unique to a scalar multiplier given by:

$$G = W_{n-m}{}^{\perp} = [w5 \;\; w1]^{\perp} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \\ -1 & 1 \end{bmatrix}^{\perp} \quad G = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \tag{33}$$

By inspection, G is:

$$G = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \tag{34}$$

Using Eq. (24), $\lambda_m = -2$ yields $M = -2$, irrespective of GW_m which is a scalar in our case. Using Eq. (17), K is determined and is equal to:

$$K = [\,5 \quad -5 \quad -3\,] \tag{35}$$

which results in the assignment of the -1, -2 and -5 eigenvalues.

Example 2:

Consider a two-input system given by:

$$\dot{x} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 3 & 1 & -2 & 2 \\ -3 & -1 & -1 & -5 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} u \tag{36}$$

Using the controllability tests, it turns out that -1 and -3 are uncontrollable eigenvalues.

It is desired to assign the two eigenvalues $-2 \pm j$, along with the uncontrollable eigenvalues -1 and -3.

Once more, the -1 and -3 eigenvalues should be reassigned through G . Otherwise, we get a non-invertible GB matrix. Furthermore, we either calculate G as the annihilating matrix of the two eigenvectors associated with the -1 and -3 eigenvalues since the system has a maximum number of uncontrollable eigenvalues, or we may choose any arbitrary G matrix such that GB is nonsingular. We make things simple and choose the most systematic option, i.e. $G = B^T$, yielding:

$$G = B^T = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \tag{37}$$

For the complex pair, we can choose any matrix of real numbers having $-2 \pm j$ as eigenvalues. One selection is $M = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$, in which case, using Eq. (17), we get K as:

$$K = \begin{bmatrix} -3.5 & 0.5 & -2 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \tag{38}$$

which results in the assignment of the four eigenvalues $-2 \pm j$,-1 and -3.

For both uncontrollable systems considered above, the K matrix is not unique even for a single input case. Therefore, if the results are to be checked by other methods, different values for K may be obtained. In fact, using our method with an alternative G results in a different K .

Example 3:

A system is randomly generated using MATLAB:

$$\dot{x} = \begin{bmatrix} 4 & 5 & -3 & 4 \\ -1 & 6 & -1 & -2 \\ 1 & 1 & 4 & 5 \\ 3 & -3 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix} u \tag{39}$$

It is required to assign repeated eigenvalues such as: -5, -5, -4 and -4.

The calculations proceed as follows (see appendix, where short format of MATLAB is used for brevity):

Using Eq. (4) with z = -1, the eigenvector needed to assign the eigenvalue -5 is:

$$w5 = [A + 5I_4]^{-1} B = [\; -0.4671 \quad -0.0338 \quad -0.0365 \quad 0.5658]^T \tag{40}$$

The second -5 eigenvalue is assigned through M.

Using Eq. (4) with z = -1, the eigenvector needed to assign the eigenvalue -4 is:

$$w4 = [A + 4I_4]^{-1} B = [\; -1.0013 \quad 0.0039 \quad -0.3822 \quad 1.2111]^T \tag{41}$$

A third eigenvector $wg4$ is needed. It is taken as a generalized eigenvector associated with the -4 eigenvalue, as shown in [15]. Symbolic MATLAB is used to do the calculations as shown in the appendix.

Giving:

$$wg4 = [-235553 \quad 16942 \quad -157964 \quad 281080]^T / \, 201243 \tag{42}$$

$$G = [\, 84282/55835, \quad -541843/167505, \quad -137096/167505, \quad 1] \tag{43}$$

And

$$K = [795719/17634, -2667827/35268, -202258/8817, 1039805/35268] \tag{44}$$

Note that MATLAB represents numbers as rational numbers when calculating symbolically.

Checking the validity of K (see appendix) confirms the assignment of, -5, -5, -4 and -4 as intended.

6. CONCLUSIONS

A conjecture has been set forth after considering the assignment of a single pole to the trivial case of a single-state system, assignment of a single pole to an nth order system, and assignment of n poles to the extreme case of a system with a nonsingular B matrix. The conjecture is then proved for the general case of non-square B matrices by solving two sets of matrix equations. The method neither distinguishes between controllable or uncontrollable systems nor between single-input or multi-input cases. The application of the method is even made simpler in certain cases of uncontrollable systems. Examples are included to demonstrate the validity of the state feedback control law.

APPENDIX

```
syms lam,
ws = inv(A - lam*eye(4))*B , % ws is too lengthy to be listed down.
wg = diff(ws, lam)           , % wg is too lengthy to be listed down.
wg4 = subs(wg, -4),
G = (null([w4  wg4  w5 ].')).',
K = inv(G*B)*(-5*G - G*A),  %  M = -5
E = jordan(A+B*K),
```

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